Final Review:

#19 on the final review sheet:

Let $f(x) = \int_0^x t\sin(t^3)\,dt$.

Compute $f^{(11)}(0)$.

→ find Maclaurin series for $f(x)$:

- $$t \cdot \sin(t^3) = t \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (t^3)^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n+4}}{(2n+1)!}$$

- $$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^x t^{6n+4}\,dt$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \left. \frac{t^{6n+5}}{6n+5} \right|_0^x$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{6n+5}}{6n+5} \quad (*)$$

→ By Taylor's theorem:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad (**)$$

→ equate the coefficients of x^{11} in $(*)$ and $(**)$:

$6n+5 = 11 \longrightarrow n = 1$

$$\overset{(*)}{\frac{(-1)^1}{3!}} \cdot \frac{1}{11} = \overset{(**)}{\frac{f^{(11)}(0)}{11!}}$$

$$\frac{11!}{11} = \frac{11 \cdot 10!}{11} = 10!$$

$$\Longleftrightarrow \quad f^{(11)}(0) = -\frac{10!}{3!}$$

#2(c) on the midterm review:

$$I = \int \left(\frac{e^{\sqrt{2}} + x^{\sqrt{2}}}{\sqrt{x}} \right) dx$$

$$= e^{\sqrt{2}} \int x^{-1/2}\, dx + \int x^{\sqrt{2}-1/2}\, dx$$

$$= e^{\sqrt{2}} \frac{x^{1/2}}{-1/2+1} + \frac{x^{\sqrt{2}+1/2}}{\sqrt{2}+1/2} + C$$

#2(d) on the midterm review:

$$I = \int \left(\frac{2}{3x} - \frac{1}{\sqrt{4-x^2}} \right) dx$$

$$= I_1 - I_2$$

$$I_1 = \frac{2}{3} \int \frac{dx}{x} = \frac{2}{3} \ln|x| + C_1$$

$$I_2 = \int \frac{dx}{\sqrt{4-x^2}} = \int \frac{dx}{2\sqrt{1-\left(\frac{x}{2}\right)^4}}$$

u-sub: $u = \frac{x}{2}$, $du = \frac{dx}{2}$

$$= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C_2$$

$$= \sin^{-1}\left(\frac{x}{2}\right) + C_2$$

$$I = I_1 - I_2 = \frac{2}{3}\ln|x| - \sin^{-1}\left(\frac{x}{2}\right) + C$$

#8 on the midterm review:

$$I = \int_2^4 (x-1)^2\,dx,$$ evaluate I using Riemann sums.

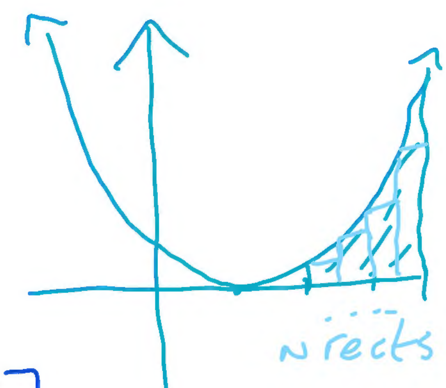
$f(x) = (x-1)^2$

→ divide the area into $N \geq 1$ rectangles:

$$\Delta x = \frac{b-a}{N} = \frac{4-2}{N} = \frac{2}{N}$$

$$x_k^* = a + k\Delta x = 2 + \frac{2k}{N}, \text{ for } k = 0, 1, 2, \dots, N$$

$$\text{let } S_N = \sum_{k=1}^{N} \Delta x\, f(x_k^*)$$



$$= \frac{2}{N}\sum_{k=1}^{N}\left(1+\frac{2k}{N}\right)^2$$

$$= \frac{2}{N}\sum_{k=1}^{N}\left[1 + \frac{4k}{N} + \frac{4k^2}{N^2}\right]$$

$$= \frac{2}{N}\left[\cancel{N}^{2} + \frac{\cancel{4}^{2}}{N}\cdot\frac{N(N+1)}{\cancel{2}} + \frac{4}{N^2}\,\frac{N(N+1)(2N+1)}{6}\right]$$

Then $I = \lim_{N \to \infty} S_N = 2\left(1 + 2 + \frac{8}{6}\right)$

$$= \frac{26}{3}$$

$\to$ check our work:

$$I = \int_2^4 (x-1)^2 \, dx = \left.\frac{(x-1)^3}{3}\right|_2^4$$

$$= \frac{3^3}{3} - \frac{1}{3} = \frac{26}{3} \checkmark$$

#12 on the midterm review sheet:

$$F(x) = \int_{\frac{8}{x}}^{x^2} \left(\frac{t}{1-\sqrt{t}}\right) dt$$

find $F'(2)$.

$$G(x) = \int_a^x \left(\frac{t}{1-\sqrt{t}}\right) dt$$

By the FTC, $G'(x) = \dfrac{x}{1-\sqrt{x}}$

By the other FTC,

$$F(x) = G(x^2) - G\left(\frac{8}{x}\right)$$

$$F'(x) = \frac{d}{dx}\left[F(x)\right] = G'(x^2)(2x) - G'\left(\frac{8}{x}\right)\left(-\frac{8}{x^2}\right)$$

$$= (2x)\frac{x^2}{1-x} + \frac{8}{x^2} \cdot \frac{8}{x} \cdot \frac{1}{1-\sqrt{8/x}} \quad (*)$$

so by $(*)$

$$F'(2) = \frac{4 \cdot 4}{-1} + \frac{64}{8} \cdot \frac{1}{1-\sqrt{4}}$$

$$= -16 + 8(-1) = -24$$

#15(a) on the midterm review:

$$I = \int \frac{1}{x^2} \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) dx$$

u-sub: $u = \frac{1}{x}$, $du = -\frac{dx}{x^2}$

$\longleftrightarrow \frac{dx}{x^2} = -du$

$$= -\int \sec(u)\tan(u)\,du$$

$$= -\sec(u) + C$$

$$= -\sec\left(\frac{1}{x}\right) + C$$

#16(b) on the midterm review:

$$I = \int \frac{e^{2x}}{\sqrt{4-3e^{2x}}}\,dx$$

u-sub: $u = 4-3e^{2x}$

$du = -6e^{2x}\,dx \longleftrightarrow e^{2x}\,dx = -\frac{1}{6}\,du$

$$= -\frac{1}{6}\int \frac{du}{\sqrt{u}} = -\frac{1}{6}\,\frac{\sqrt{u}}{1/2} + C$$

$$= -\frac{1}{3}\sqrt{u} + C = -\frac{1}{3}\sqrt{4-3e^{2x}} + C$$

#22(d) on the midterm review:

$$I = \int x^3 e^{x^2}\,dx$$

u-sub: $u = x^2$, $du = 2x\,dx$

$x^3\,dx = \frac{1}{2}u\,du$

$$= \frac{1}{2}\int u e^u\,du$$

$$= \frac{1}{2}\left[ue^u - \int e^u\,du\right]$$

$$= \frac{1}{2}ue^u - \frac{1}{2}e^u + C$$

$$= \frac{1}{2}\left(x^2e^{x^2} - e^{x^2}\right) + C$$

IBP:
- $\int u\,dv = uv - \int v\,du$
- how to choose u?

ILATE

$\downarrow$ u $\quad$ $\hookrightarrow e^u$

- $u = u$ $\qquad$ $dv = e^u\,du$

$du = du$ $\qquad$ $v = e^u$

#23(b) on the midterm review:

$$I = \int \sin^5(2x)\cos^3(2x)\,dx$$

$$\left[\cos^3(2x) = \cos(2x)\left(1-\sin^2(2x)\right)\right]$$

$$u = \sin(2x),\quad du = 2\cos(2x)\,dx$$

$$= \frac{1}{2}\int u^5(1-u^2)\,du = \frac{1}{2}\left(\frac{u^6}{6} - \frac{u^8}{8}\right) + C$$

$$= \frac{\sin^6(2x)}{12} - \frac{\sin^8(2x)}{16} + C$$